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THE DIRAC EQUATION APPROACH TO SPIN-$\frac{1}{2}$ PARTICLE BEAM OPTICS

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The traditional approach to accelerator optics, based mainly on classical mechanics, is working excellently from the practical point of view. However, from the point of view of curiosity, as well as with a view to explore quantitatively the consequences of possible small quantum corrections to the classical theory, a quantum mechanical formalism of accelerator optics for the Dirac particle is being developed recently. Here, the essential features of such a quantum beam optical formalism for any spin-$\frac{1}{2}$ particle are reviewed. It seems that the quantum corrections, particularly those due to the Heisenberg uncertainty, could be important in understanding the nonlinear dynamics of accelerator beams.

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1 Introduction

Why should one bother about a quantum mechanical treatment of accelerator *optics* when the classical treatment works so well? This is a natural question indeed. There is no *prima facie* reason at all to believe that a quantum mechanical treatment would be necessary to deal with any aspect of accelerator *optics* design. As has been rightly pointed out [1], primary effects in conventional accelerators are essentially classical since the de Broglie wavelength of the high energy beam particle is much too small compared to the typical apertures and the energy radiated is typically low and of long wavelength. However, it is being slowly recognized that quantum effects are still important due to demands on high precision in accelerator performance and ever increasing demands for higher beam energy, luminosity and brightness [1]. Also, there is a growing feeling now that a complete picture of spin polarization can only be obtained on the basis of coupled spin $\leftrightarrow$ orbit phase-space transport equations (with and without radiation) and to include all the subtleties of radiation one has to begin with quantum mechanics since classical white noise models simply would not suffice for all situations [2]. I like to add:

- After all, accelerator beam is a quantum mechanical system and one may be curious to know how its classical behavior can be understood, in detail, from a quantum mechanical formalism based on the appropriate relativistic wave equation.

- As has been revealed by recent studies [3] the passage from quantum theory to classical theory is not a simple affair, particularly when the system is a complicated nonlinear dynamical system with regular and chaotic regions in its phase-space. Since accelerator beams are such systems [4] it is time that the quantum mechanics of accelerator optics is looked at seriously.

Essentially from the point of view of curiosity, the axially symmetric magnetic lens was first studied [5] based completely on the Dirac equation. Later works [6, 7] led to further insights into the quantum mechanics of spin-$\frac{1}{2}$ particle beam optics. Quantum mechanics of the Klein-Gordon (spin-0) and nonrelativistic Schrödinger charged-particle beams were also studied [7, 8]. These works dealt essentially with aspects of ion optics and electron optical imaging (for an excellent survey of scalar electron wave optics see the third volume of the encyclopaedic three-volume text book of Hawkes & Kasper [9]; this contains also references to earlier works on the use of the Dirac equation in electron wave optics problems like diffraction, to take into account the spinor nature of the electron).

In the context of accelerator physics also, like in the case of electron and ion beam optical device technologies, the practice of design of beam optical elements is based mainly on classical physics. As is well known, various aspects

of accelerator beam dynamics like orbital motion, spin evolution and beam polarization, radiation and quantum fluctuations of trajectories, are analyzed piecewise using classical, semiclassical, or quantum theories, or a mixture of them, depending on the situation treated. Quantum mechanical implications for low energy polarized (anti)proton beams in a spin-splitter device, using the transverse Stern-Gerlach (SG) kicks, have been analyzed [10] on the basis of nonrelativistic Schrödinger equation.

To obtain the coupled spin $\leftrightarrow$ phase-space transport equations, for the spin-$\frac{1}{2}$ particle, one needs an appropriate quantum Hamiltonian. Such a Hamiltonian was stated, as following from the systematic Foldy-Wouthuysen (FW) transformation technique [11], by Derbenev and Kondratenko [12] in 1973 as the starting point of their radiation calculations but no explicit construction was given (such a Hamiltonian can also be justified [13] using the Pauli reduction of the Dirac theory). The Derbenev-Kondratenko (DK) Hamiltonian has been used [14] to construct a completely classical approach to beam optics, including spin components as classical variables. Now, a detailed derivation of the DK Hamiltonian has been given [2] and a completely quantum mechanical formalism is being developed [2] in terms of the 'machine coordinates' and 'observables', suitable for treating the physics of spin-$\frac{1}{2}$ polarized beams from the point of view of machine design.

Independent of the DK formalism, recently [15, 16] we have made a beginning in the application of the formalism of the Dirac spinor beam optics, developed earlier ([5]-[7]) mostly with reference to electron microscopy, to accelerator optics to understand in a unified way the orbital motion, SG kicks, and the Thomas-Bargmann-Michel-Telegdi (TBMT) spin evolution. Here, I present the essential features of our work, done so far, on the quantum beam optical approach to accelerator optics of spin-$\frac{1}{2}$ particles based on the Dirac equation.

2 Quantum beam optics of the Dirac particle

Our formalism of quantum beam optics of the Dirac particle, in the context of accelerators, is only in the beginning stages of development. So, naturally there are several simplifying assumptions: We deal only with the single particle dynamics, based on the single particle interpretation of the Dirac equation, ignoring all the inter-particle interactions and statistical aspects. Only monoenergetic beam is considered. The treatment is at the level of paraxial approximation, so far, though the general framework of the theory is such that extension to the case of nonparaxial (nonlinear) systems is straightforward. Only time-independent magnetic optical elements with straight axis are considered. Electromagnetic field is treated as classical. And, radiation is ignored.

Thus, we are dealing with elastic scattering of the particles of a monoenergetic beam by an optical element with a straight axis along the z-direction and comprising a static magnetic field $\boldsymbol{B} = \mathrm{curl}\,\boldsymbol{A}$. Hence, the 4-component

spinor wavefunction of the beam particle can be assumed to be of the form $\Psi(\boldsymbol{r},t) = \psi(\boldsymbol{r})\exp(-iEt/\hbar)$, where E is the total (positive) energy of the particle of mass m, charge q, and anomalous magnetic moment μ_a. The spatial part of the wavefunction $\psi(\boldsymbol{r})$ has to obey the time-independent Dirac equation

$$H\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r})\,, \tag{1}$$

where the Hamiltonian H, including the Pauli term, is given by

$$\begin{gathered}
H = \beta mc^2 + c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{\pi}} - \mu_a\beta\boldsymbol{\Sigma}\ \cdot\boldsymbol{B}\,, \\
\beta = \begin{pmatrix} \mathbb{1} & \boldsymbol{0} \\ \boldsymbol{0} & -\mathbb{1} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \boldsymbol{0} \end{pmatrix}, \quad \boldsymbol{\Sigma}\ = \begin{pmatrix} \boldsymbol{\sigma} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\sigma} \end{pmatrix}, \\
\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \boldsymbol{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{gathered} \tag{2}$$

with $\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{p}} - q\boldsymbol{A} = -i\hbar\boldsymbol{\nabla} - q\boldsymbol{A}$. Let p be the design momentum of the beam particle along the $+z$-direction so that $E = +\sqrt{m^2c^4 + c^2p^2}$. The beam is assumed to be paraxial: $|\boldsymbol{p}_\perp| \ll |\boldsymbol{p}| = p$ and $p_z > 0$.

Since we are interested in studying the propagation of the beam along the $+z$-direction we would like to rewrite Eq. (1) as

$$i\hbar\frac{\partial}{\partial z}\psi(\boldsymbol{r}_\perp;z) = \mathcal{H}\psi(\boldsymbol{r}_\perp;z)\,. \tag{3}$$

So, we multiply Eq. (1) from left by α_z/c and rearrange the terms to get

$$\begin{aligned}
\mathcal{H} &= -p\beta\chi\alpha_z - qA_z\mathbb{1} + \alpha_z\boldsymbol{\alpha}_\perp\cdot\hat{\boldsymbol{\pi}}_\perp + (\mu_a/c)\beta\alpha_z\boldsymbol{\Sigma}\ \cdot\boldsymbol{B}\,, \\
\chi &= \begin{pmatrix} \xi\mathbb{1} & \boldsymbol{0} \\ \boldsymbol{0} & -\xi^{-1}\mathbb{1} \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} \mathbb{1} & \boldsymbol{0} \\ \boldsymbol{0} & \mathbb{1} \end{pmatrix}, \quad \xi = \sqrt{\frac{E+mc^2}{E-mc^2}}\,.
\end{aligned} \tag{4}$$

Note that the matrix $\beta\chi\alpha_z$, coefficient of $-p$ in $\mathcal{H}$, is diagonalized as follows:

$$M(\beta\chi\alpha_z)M^{-1} = \beta\,, \qquad M = \frac{1}{\sqrt{2}}(\mathbb{1} + \chi\alpha_z)\,. \tag{5}$$

Hence, let us define

$$\psi' = M\psi\,. \tag{6}$$

This turns Eq. (3) into

$$i\hbar\frac{\partial}{\partial z}\psi' = \mathcal{H}'\psi'\,, \qquad \mathcal{H}' = M\mathcal{H}M^{-1} = -p\beta + \mathcal{E} + \mathcal{O}\,, \tag{7}$$

with the 'even' operator $\mathcal{E}$ and the 'odd' operator $\mathcal{O}$ given, respectively, by the diagonal and off-diagonal parts of

$$\mathcal{E}+\mathcal{O} = \left(\begin{array}{ll} \begin{array}{l} \left[-qA_z\mathbb{1} - (\mu_a/2c)\right. \\ \times\left\{\left(\xi+\xi^{-1}\right)\boldsymbol{\sigma}_\perp\cdot\boldsymbol{B}_\perp\right. \\ \left.\left.+\left(\xi-\xi^{-1}\right)\sigma_z B_z\right\}\right] \end{array} & \begin{array}{l} \xi\left[\boldsymbol{\sigma}_\perp\cdot\hat{\boldsymbol{\pi}}_\perp - (\mu_a/2c)\right. \\ \times\left\{i\left(\xi-\xi^{-1}\right)\right. \\ (B_x\sigma_y - B_y\sigma_x) \\ \left.\left.-\left(\xi+\xi^{-1}\right)B_z\mathbb{1}\right\}\right] \end{array} \\ \begin{array}{l} -\xi^{-1}\left[\boldsymbol{\sigma}_\perp\cdot\hat{\boldsymbol{\pi}}_\perp + (\mu_a/2c)\right. \\ \times\left\{i\left(\xi-\xi^{-1}\right)\right. \\ (B_x\sigma_y - B_y\sigma_x) \\ \left.\left.+\left(\xi+\xi^{-1}\right)B_z\mathbb{1}\right\}\right] \end{array} & \begin{array}{l} \left[-qA_z\mathbb{1} - (\mu_a/2c)\right. \\ \times\left\{\left(\xi+\xi^{-1}\right)\boldsymbol{\sigma}_\perp\cdot\boldsymbol{B}_\perp\right. \\ \left.\left.-\left(\xi-\xi^{-1}\right)\sigma_z B_z\right\}\right] \end{array} \end{array}\right). \quad (8)$$

The significance of the transformation in Eq. (6) is that for a paraxial beam propagating in the forward $(+z)$ direction ψ' is such that its lower pair of components are very small compared to the upper pair of components, exactly like for a positive energy nonrelativistic Dirac spinor. Note the perfect analogy: nonrelativistic case $\rightarrow$ paraxial, positive energy $\rightarrow$ forward propagation, and $mc^2 \rightarrow -p$ in Eq. (2) and Eq. (7) respectively.

Let us recall that the FW transformation technique [11, 17] is the most systematic way of analyzing the standard Dirac equation as a sum of the nonrelativistic part and a series of relativistic correction terms. So, the application of an analogous technique to Eq. (7) should help us analyze it as a sum of the paraxial part and a series of nonparaxial correction terms. To this end, we define the FW-like transformation

$$\psi_1 = S_1\psi', \qquad S_1 = \exp\left(-\beta\mathcal{O}/2p\right). \quad (9)$$

The resulting equation for ψ_1 is

$$\begin{aligned} & i\hbar\frac{\partial}{\partial z}\psi_1 = \mathcal{H}_1\psi_1, \quad \mathcal{H}_1 = S_1\mathcal{H}'S_1^{-1} - i\hbar S_1\frac{\partial}{\partial z}\left\{S_1^{-1}\right\} = -p\beta + \mathcal{E}_1 + \mathcal{O}_1, \\ & \mathcal{E}_1 = \mathcal{E} - \frac{1}{2p}\beta\mathcal{O}^2 + \cdots, \quad \mathcal{O}_1 = -\frac{1}{2p}\beta\left\{[\mathcal{O},\mathcal{E}] + i\hbar\frac{\partial}{\partial z}\mathcal{O}\right\} + \cdots. \end{aligned} \quad (10)$$

A series of such transformations successively with the same type of recipe as in Eq. (9) eliminates the odd parts from $\mathcal{H}'$ up to any desired order in $1/p$. It should also be mentioned that these FW-like transformations preserve the property of ψ' that its upper pair of components are large compared to the lower pair of components. We shall stop with the above first step which would correspond to the paraxial, or the first order, approximation.

Since the lower pair of components of ψ_1 are almost vanishing compared to its upper pair of components and the odd part of $\mathcal{H}_1$ is negligible compared to its even part, up to the first order approximation we are considering, we can effectively introduce a two-component spinor formalism based on the representation of Eq. (10). Naming the two-component spinor comprising the upper

pair of components of ψ_1 as ψ'' and calling the 2×2 11-block element of $\mathcal{H}_1$ as $\mathcal{H}''$ it is clear from Eq. (10) that we can write

$$
\begin{aligned}
& i\hbar \frac{\partial}{\partial z} \psi'' = \mathcal{H}'' \psi'' \,, \\
\mathcal{H}'' &\approx \left(-p - qA_z + \frac{1}{2p} \hat{\pi}_{\perp}^2 \right) - \frac{1}{p} \left\{ (q+\epsilon) B_z S_z + \gamma \epsilon \boldsymbol{B}_{\perp} \cdot \boldsymbol{S}_{\perp} \right\} \,, \\
\text{with} \quad & \hat{\pi}_{\perp}^2 = \hat{\pi}_x^2 + \hat{\pi}_y^2 \,, \quad \epsilon = 2m\mu_a/\hbar \,, \quad \gamma = E/mc^2 \,, \quad \boldsymbol{S} = \hbar \boldsymbol{\sigma}/2. \qquad (11)
\end{aligned}
$$

Up to now, all observables, field components, etc., are defined with reference to the laboratory frame. But, as is usual in accelerator physics, we have to define spin with reference to the instantaneous rest frame of the particle while keeping the other observables, field components, etc., defined with reference to the laboratory frame. For this, we have to transform ψ'' further to an 'accelerator optics representation', say, $\psi_A = T_A \psi''$. The choice of T_A is dictated by the following consideration. Let the operator $\hat{O}$ correspond to an observable O in the Dirac representation (Eq. (2) or Eq. (3)). The operator corresponding to O in the representation of Eq. (11) can be taken to be given by

$$
\hat{O}'' = \text{the hermitian part of the } 11-\text{block element of } \left(S_1 M \hat{O} M^{-1} S_1^{-1} \right) . \qquad (12)
$$

The corresponding operator in the accelerator optics representation will be

$$
\hat{O}_A = \text{the hermitian part of } \left(T_A \hat{O}'' T_A^{-1} \right) . \qquad (13)
$$

The operator

$$
\boldsymbol{S}^{(R)} = \frac{1}{2}\hbar \begin{pmatrix} \boldsymbol{\sigma} - \frac{c^2(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}}\hat{\boldsymbol{\pi}} + \hat{\boldsymbol{\pi}}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})}{2E(E+mc^2)} & \frac{c\hat{\boldsymbol{\pi}}}{E} \\ \frac{c\hat{\boldsymbol{\pi}}}{E} & -\boldsymbol{\sigma} + \frac{c^2(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}}\hat{\boldsymbol{\pi}} + \hat{\boldsymbol{\pi}}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})}{2E(E+mc^2)} \end{pmatrix} . \qquad (14)
$$

corresponds to the rest-frame spin in the Dirac representation [18]. We demand that the components of the rest-frame spin operator in the accelerator optics representation be simply the Pauli spin matrices, *i.e.*, $\boldsymbol{S}_A^{(R)} \approx \hbar\boldsymbol{\sigma}/2$, up to the first order (paraxial) approximation. This demand leads to the choice

$$
\psi_A = T_A \psi'' \,, \qquad T_A = \exp\left\{ -i \left(\hat{\pi}_x \sigma_y - \hat{\pi}_y \sigma_x \right) / 2p \right\} . \qquad (15)
$$

Now, finally, with the transformation given by Eq. (15), the desired basic equation of the quantum beam optics of the Dirac particle becomes, up to paraxial approximation,

$$
\begin{aligned}
i\hbar \frac{\partial}{\partial z} \psi_A = \mathcal{H}_A \psi_A \,, \quad & \mathcal{H}_A \approx \left(-p - qA_z + \frac{1}{2p} \hat{\pi}_{\perp}^2 \right) + \frac{\gamma m}{p} \underline{\boldsymbol{\Omega}} \cdot \boldsymbol{S} \,, \\
& \text{with } \; \underline{\boldsymbol{\Omega}} = -\frac{1}{\gamma m} \left\{ q\boldsymbol{B} + \epsilon \left(\boldsymbol{B}_{\parallel} + \gamma \boldsymbol{B}_{\perp} \right) \right\} . \qquad (16)
\end{aligned}
$$

Here, $\boldsymbol{B}_{\parallel}$ and $\boldsymbol{B}_{\perp}$ are the components of $\boldsymbol{B}$ in the $+z$-direction (the predominant direction of motion of the beam particles) and perpendicular to it, unlike in the usual TBMT vector $\boldsymbol{\Omega}$ in which the components $\boldsymbol{B}_{\parallel}$ and $\boldsymbol{B}_{\perp}$ are defined with respect to the direction of the instantaneous velocity of the particle. The quantum beam optical Hamiltonian $\mathcal{H}_A$ is the beam optical version of the DK Hamiltonian in the paraxial approximation. To get the higher order corrections, in terms of $\hat{\boldsymbol{\pi}}_{\perp}/p$ and $\hbar$, we have to go beyond the first FW-like transformation.

It must be noted that while the exact quantum beam optical Dirac Hamiltonian $\mathcal{H}$ (see Eq. (3)) is nonhermitian the nonunitary FW-like transformation has projected out a hermitian $\mathcal{H}_A$ (see Eq. (16)). Thus, for the particle that survives the transport through the system, without getting scattered far away, ψ_A has unitary evolution along the z-axis. Hence, we can normalize the two-component ψ_A, at any z, as $\langle\psi_A(z)\,|\psi_A(z)\rangle = \int\int dxdy\,\psi_A^{\dagger}\psi_A = 1$. This normalization will be conserved along the optic (z) axis. Then, for any observable O represented by a hermitian operator $\hat{O}_A$, in the accelerator optics representation (Eq. (16)), we can define the average at the transverse plane at any z as

$$\langle O\rangle(z) = \langle\psi_A(z)|\,\hat{O}_A\,|\psi_A(z)\rangle = \int\int dxdy\,\psi_A^{\dagger}\hat{O}_A\psi_A\,. \tag{17}$$

Now, studying the z-evolution of $\langle O\rangle(z)$ is straightforward. Integration of Eq. (16) gives

$$\psi_A(z') = \hat{U}(z',z)\psi_A(z)\,, \tag{18}$$

where the unitary z-evolution operator $\hat{U}(z',z)$ can be obtained by the standard quantum mechanical methods. Thus, the relations

$$\langle O\rangle(z') = \langle\psi_A(z)|\,\hat{U}(z',z)^{\dagger}\hat{O}_A\hat{U}(z',z)\,|\psi_A(z)\rangle\ , \tag{19}$$

for the relevant set of observables, give the transfer maps for the quantum averages (or their classical values *à la* Ehrenfest) from the plane at z to the plane at z'. In the classical limit this relation (Eq. (19)) becomes the basis for the Lie algebraic approach to classical beam optics [19]. The Lie algebraic approach has been studied [20] in the context of spin transfer map also using the classical formalism.

The main problem of accelerator optics is to know the transfer maps for the quantum averages of the components of position, momentum, and the rest-frame spin between transverse planes containing the optical elements. We have already seen that the rest-frame spin is represented in the accelerator optics representation by the Pauli spin matrices. Let us take that the observed position of the Dirac particle corresponds to the mean position operator of the FW theory [11] or what is same as the Newton-Wigner position operator [21]. Then, one can show, using Eq. (12) and Eq. (13), that in the accelerator optics representation the transverse position operator is given by $\boldsymbol{r}_{\perp}$ up to the first order approximation (details are given elsewhere [22]). For the transverse momentum in free space the operator is $\hat{\boldsymbol{p}}_{\perp}$ in the accelerator optics representation.

3 An example: the normal magnetic quadrupole lens

For an ideal normal magnetic quadrupole lens of length L comprising the field $\boldsymbol{B} = (Gy, Gx, 0)$, corresponding to $\boldsymbol{A} = \left(0, 0, \frac{1}{2}G\left(y^2 - x^2\right)\right)$, and bounded by transverse planes at z_i and $z_f = z_i + L$, the quantum accelerator optical Hamiltonian (Eq. (16)) becomes

$$\mathcal{H}_A = \begin{cases} -p + \frac{1}{2p}\hat{p}_\perp^2 \,, \quad \text{for} \quad z < z_i \quad \text{and} \quad z > z_f \,, \\ -p + \frac{1}{2}qG\left(x^2 - y^2\right) + \frac{1}{2p}\hat{p}_\perp^2 - \frac{(q+\gamma\epsilon)G\hbar}{2p}\left(y\sigma_x + x\sigma_y\right) \,, \\ \qquad\qquad \text{for} \quad z_i \leq z \leq z_f \,. \end{cases} \tag{20}$$

Note that the choice of $\boldsymbol{A}$ in a different gauge will not affect the average values defined by Eq. (17). Now, using the formalism of the previous section, it is straightforward to find the desired transfer maps in this case (details are found elsewhere [15, 22]). The results are: with $\eta = (q + \gamma\epsilon)GL\hbar/2p^2$, $K = \sqrt{qG/p}$, $\lambda = h/p$, $\langle\ \rangle_i = \langle\ \rangle(z_i)$, and $\langle\ \rangle_f = \langle\ \rangle(z_f)$,

$$\begin{aligned}
\begin{pmatrix} \langle x\rangle_f \\ \langle\hat{p}_x\rangle_f \end{pmatrix} &\approx \begin{pmatrix} \cos KL & \frac{1}{pK}\sin KL \\ -pK\sin KL & \cos KL \end{pmatrix} \\
&\qquad \times \left(\begin{pmatrix} \langle x\rangle_i \\ \langle\hat{p}_x\rangle_i \end{pmatrix} + \eta \begin{pmatrix} (\cos KL - 1)\left\langle\sigma_y\right\rangle_i/K^2L \\ -\left(p\sin KL\right)\left\langle\sigma_y\right\rangle_i/KL \end{pmatrix} \right) , \\
\begin{pmatrix} \langle y\rangle_f \\ \langle\hat{p}_y\rangle_f \end{pmatrix} &\approx \begin{pmatrix} \cosh KL & \frac{1}{pK}\sinh KL \\ pK\sinh KL & \cosh KL \end{pmatrix} \\
&\qquad \times \left(\begin{pmatrix} \langle y\rangle_i \\ \langle\hat{p}_y\rangle_i \end{pmatrix} + \eta \begin{pmatrix} -(\cosh KL - 1)\left\langle\sigma_x\right\rangle_i/K^2L \\ -\left(p\sinh KL\right)\left\langle\sigma_x\right\rangle_i/KL \end{pmatrix} \right) , \\
\left\langle S_x\right\rangle_f &\approx \left\langle S_x\right\rangle_i + \frac{4\pi\eta}{\lambda}\left(\left(\frac{\sin KL}{KL}\right)\left\langle xS_z\right\rangle_i + \left(\frac{\cos KL - 1}{K^2Lp}\right)\left\langle\hat{p}_xS_z\right\rangle_i \right) , \\
\left\langle S_y\right\rangle_f &\approx \left\langle S_y\right\rangle_i - \frac{4\pi\eta}{\lambda}\left(\left(\frac{\sinh KL}{KL}\right)\left\langle yS_z\right\rangle_i - \left(\frac{\cosh KL - 1}{K^2Lp}\right)\left\langle\hat{p}_yS_z\right\rangle_i \right) , \\
\left\langle S_z\right\rangle_f &\approx \left\langle S_z\right\rangle_i - \frac{4\pi\eta}{\lambda}\left\{ \left(\frac{\sin KL}{KL}\right)\left\langle xS_x\right\rangle_i - \left(\frac{\sinh KL}{KL}\right)\left\langle yS_y\right\rangle_i \right. \\
&\quad \left. + \left(\frac{\cos KL - 1}{K^2Lp}\right)\left\langle\hat{p}_xS_x\right\rangle_i + \left(\frac{\cosh KL - 1}{K^2Lp}\right)\left\langle\hat{p}_yS_y\right\rangle_i \right\} .
\end{aligned} \tag{21}$$

Obviously we have obtained the well known classical transfer maps (matrices) for the transverse phase-space coordinates, and more, *i.e.*, the transverse SG kicks [10, 23]. The longitudinal SG kick, which has been proposed [23] as a better alternative to the transverse SG kicks for making a spin-splitter device to produce polarized (anti)proton beams, can also be understood [15, 22] using the present quantum beam optical formalism. The skew magnetic quadrupole can also be analyzed [24] in the same way as here.

4 Conclusion

In summary, we have seen how one can obtain a fully quantum mechanical formalism of the accelerator beam optics for a spin-$\frac{1}{2}$ particle, with anomalous magnetic moment, starting *ab initio* from the Dirac-Pauli equation. This formalism leads naturally to a unified picture of orbital and spin dynamics taking into account the effects of the Lorentz force, the SG force and the TBMT equation for spin evolution. Only the lowest order (paraxial) approximation has been considered in some detail, with an example. It is clear from the general theory, presented briefly here, that the approach is suitable for handling any magnetic optical element with straight axis and computations can be carried out to any order of accuracy by easily extending the order of approximation. It should be emphasized that the present formalism is valid for all values of design momentum p from the nonrelativistic case to the ultrarelativistic case. The approximation scheme is based only on the fact that for a beam, constituted by particles moving predominantly in one direction, the transverse kinetic momentum is very small compared to the longitudinal kinetic momentum.

We hope to address elsewhere [22] some of the issues related to the construction of a more general theory overcoming the limitations of the present formalism. With reference to the inclusion of multiparticle dynamics within the present formalism, it might be profitable to be guided by the so-called thermal wave model which has been extensively developed [25] in recent years to account for the classical collective behavior of a charged-particle beam, by associating with the classical beam a quantum-*like* wavefunction obeying a Schrödinger-*like* equation with the role of $\hbar$ played by the beam emittance ε.

To conclude, let me emphasize the significance of the quantum formalism for beam optics. The following question has been raised: What are the uses of quantum formalisms in beam physics? [1] Of course, as we have seen above, we understand the quantum mechanics underlying the observed classical behavior of the beam: when the $\hbar$-dependent quantum corrections are worked out through the higher order FW-like transformations it is found that they are really negligible. In my opinion, quantum formalism of beam optics has more significant uses. To see this, let me cite the following cases: (1) Recently there is a renewed interest [26] in the form of the force experienced by a spinning relativistic particle in an external electromagnetic field. Such studies are particularly important in evaluating the possible mechanisms of spin-splitter devices [23]. A thorough analysis based on general Poincaré covariance, up to first order in spin, shows that classical spin-orbit systems can be characterized [27], at best, by five phenomenological parameters. This just points to the fact that spin being essentially a quantum aspect one must eventually have a quantum formalism to understand really the high energy polarized accelerator beams. (2) A look at Eq. (17) shows that the form of the transfer map for the quantum averages of observables will differ from the form of the corresponding classical maps by terms of the type $\langle f(\hat{O}_1, \hat{O}_2, \ldots,)\rangle - f(\langle\hat{O}_1\rangle, \langle\hat{O}_2\rangle, \ldots,) \neq 0$; the differences are

essentially due to the quantum uncertainties associated with ψ_A at the initial z_i. For example, a term like x^3 in a classical map will become in the corresponding quantum map $\langle x^3 \rangle = \langle x \rangle^3 + 3\langle x \rangle \langle (x - \langle x \rangle)^2 \rangle + \langle (x - \langle x \rangle)^3 \rangle$ which need not vanish even on the axis (where $\langle x \rangle = 0$). It is thus clear that, essentially, quantum mechanics modifies the coefficients of the various linear and nonlinear terms in the classical map making them dependent on the quantum uncertainties associated with the wavefunction of the input beam; actually, even terms absent in the classical map will be generated in this manner with coefficients dependent on the quantum uncertainties as a result of modifications of the other terms (a related idea of fuzzy classical mechanics, or fuzzy quantum evolution equations, occurs in a different context [28]). This quantum effect could be significant in the nonlinear dynamics of accelerator beams. This effect is relevant for spin dynamics too. For example, it is seen in Eq. (21) that the spin transfer map is not linear in its components, in principle, even in the lowest order approximation, since terms of the type, say, $\langle x S_z \rangle_i$, $\langle \hat{p}_x S_z \rangle_i$, etc., are not the same as $\langle x \rangle_i \langle S_z \rangle_i$, $\langle \hat{p}_x \rangle_i \langle S_z \rangle_i$, etc., respectively, in general.

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